OPTIMALLY UNBIASED MA ESTIMATING OF CLOCK TIME ERRORS IN GPS-BASED TIMEKEEPING

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ABSTRACT

A new optimally unbiased moving average (MA) filter is examined being intended for the time error online estimating and synchronization in GPS-based timekeeping. The approach proceeds from the slowly changing nature of a time error generated by a local clock assuming its function to be linear for the number *N* of points in the average. The filter and its improved modification produces an unbiased estimate with two times more noise than a simple MA for *N* >> 1. Both the numerical simulation and the filtering of the real GPS-based time error processes are provided and discussed in detail.

1. INTRODUCTION

Timekeeping itself solves two principal tasks, namely estimating the time error of a local clock using the reference timing signals (usually of GPS) and its elimination by a synchronization loop. Certainly, optimal linear stochastic estimation theory [1], named Kalman filtering theory, solves these tasks straightforward. Even so, the answer is not so explicit in terms of the mean-error (bias) and variance (noise). The alternative approach, called moving averaging (MA), is still used in test and measurement [2], since of all the possible linear filters that could be used, including the Kalman, a simple MA produces the lowest noise [3], which is the input noise reduced by the square-root of the number of points *N* in the average. The problem here is the readily seen estimate bias, unlike the Kalman case [4], the absence of this bias would make the simple MA the best.

The MA bias was recently efficiently reduced in [5] and the produced optimally unbiased filter has showed good performance as compared with the Kalman filter [6] being applied to the time error model [7] and [8]

$$x_n = x_0 + y_0 \Delta n + \frac{D}{2}\Delta^2 n^2 + w_{xn}\,, \tag{1}$$

where $n = 0,\ 1,\ \ldots;\ \Delta = t_n - t_{n-1}$ is the sample time; t_n is the discrete time, x_0 is the initial time error, y_0 is the initial fractional frequency offset of a local clock from the reference frequency, *D* is the linear fractional frequency drift rate, and $w_{xn}(t)$ is the random time error deviation component. Noting the special feature of the time error (1), that is its ability to change slowly and rather linearly (owing to a small rate *D*) during the averaging interval $\theta = \Delta(N-1)$, we assumed in [5] that the task may be solved alternatively by the MA bias elimination with the aid of stochastic approximation. In this paper we first present the new optimally unbiased filter, and then study them for the simulated data and several practical situations.

2. PROBLEM FORMULATION

Consider the discrete-time GPS-based noisy time error ξ_n (observation) given for the limited time interval of the discrete points $n - N + 1, \ldots, n$. The observation is an additive sum of a time error (1) itself and a zero-mean white Gaussian noise w_n of the GPS timing signals [9] with known constant variance σ_w^2, this is,

$$\xi_n = x_n + w_n\,, \tag{2}$$

where x_n is assumed to change linearly in the averaging interval $n - N + 1, \ldots, n$ and the noise $w_{xn}(t)$ in (1) is supposed to be much smaller than the GPS noise w_n in (2). Now estimate time error with a simple MA

$$\hat{x}'_n = \frac{1}{N}\sum_{i=0}^{N-1} \xi_{n-i} \tag{3}$$

and note that this filter with a constant weight 1/*N* inherently produces the bias (Fig. 1)

$$\Delta x_n = \hat{x}'_n - x_n \cong x_{n-N+1} - \hat{x}'_n\,. \tag{4}$$

Also note that the bias is almost negligible being related to the center of the averaging interval. Then pose the task: *Find an additional weighting function* $W'_i(N)$, $i = 0,\ 1,\ \ldots N-1$ *for the simple MA (3), aiming to eliminate the bias (4) by means of* $\hat{x}_n = \frac{1}{N}\sum_{i=0}^{N-1} W'_i(N)\xi_{n-i}$ *and provide the new unbiased estimate in form of the MA model*

$$\hat{x}_n = \sum_{i=0}^{N-1} W_i(N)\xi_{n-i}\,, \tag{5}$$

where $W_i(N) = W'_i(N)/N$ *is the required weight.*

3. WEIGHTING FUNCTION

So long as the time error function is assumed to be linear for the time span of *N* points then the bias may be formally compensated for (3) in the following way

$$\hat{x}_n = \hat{x}'_n - \Delta x_n \cong \hat{x}'_n - (x_{n-N+1} - \hat{x}'_n) = 2\hat{x}'_n - x_{n-N+1}\,. \tag{6}$$

where the unknown variable x_{n-N+1} is the time error at the starting point. Certainly, we do not know x_{n-N+1} and must find some way to evaluate it as accurately as possible. To provide this, let us recall that a nonparametric linear regression [10] serves in statistics as an optimal stochastic approximation in a sense of least mean squares (LMS) [11]. Apply the linear regression function for the averaging interval $n-N+1,...,n$, then get

$$\lambda(t) = a_n + b_n(t - c_n)\,, \tag{7}$$

where a_n, b_n, and c_n are coefficients calculated as

$$a_n = \frac{1}{N}\sum_{i=0}^{N-1}\xi_{n-i} = \hat{x}'_n\,, \tag{8}$$

$$b_n = \frac{Cov(\xi_n, t_n)}{\sigma_{tn}^2}\,, \tag{9}$$

$$c_n = \frac{1}{N}\sum_{i=0}^{N-1} t_{n-i}\,, \tag{10}$$

where $Cov(\xi_n, t_n)$ is the sample covariance of ξ and t, and σ_{tn}^2 is the sample variance of time. Since (7) stochastically approximates each point of the process in the optimal way, then its value $\lambda(t_{n-N+1})$ may be treated as the most accurate evidence of x_{n-N+1}, and then substituting $\lambda(t_{n-N+1})$ for (6) should yield the desired optimally unbiased estimate.

3.1. Optimally Unbiased Weight

Examine the above-mentioned proposition, writing

$$x_{n-N+1} \cong \lambda(t_{n-N+1})\,. \tag{11}$$

Substituting (8) and (11) for (6) yields the formula

$$\hat{x}_n = \hat{x}'_n - b_n(t_{n-N+1} - c_n)\,, \tag{12}$$

for which, first, calculate using equation (10)

$$c_n = \frac{1}{N}\sum_{i=0}^{N-1} t_{n-i} = \frac{t_{n-N+1} + t_n}{2} = \Delta\left(n - \frac{N-1}{2}\right), \tag{13}$$

then a routine transform produces the variance

$$\sigma_{tn}^2 = \frac{1}{N}\sum_{i=0}^{N-1}(t_{n-i} - c_n)^2 = \Delta^2\frac{N^2 - k}{12}\,, \tag{14}$$

where the coefficient is $k = 1$, and this leads to the covariance of ξ and t, this is

$$Cov(\xi_n, t_n) = \frac{\Delta}{N}\sum_{i=0}^{N-1}\left(\frac{N-1}{2} - i\right)\xi_{n-i}\,. \tag{15}$$

Substitute (9), (13)—(15) for (12), make the transformation, and get the desired unbiased estimate[1]

$$\hat{x}_n = \sum_{i=0}^{N-1}\frac{2(2N-1) - 6i}{N(N+1)}\xi_{n-i}\,, \tag{16}$$

[1] The same formula (16) appears if one just converts the linear regression value $\lambda(t_n)$ to the MA model (5)

Then write the required weighting function for (5) as

$$W_i(N) = \begin{cases} \dfrac{2(2N-1) - 6i}{N(N+1)}, & 0 \le i \le N-1\,, \\ 0, & otherwise \end{cases} \tag{17}$$

which is nonzero for the averaging interval only. Figure 1 sketches the shape of the weight (17), which analysis shows that, in contrast to the normal kernel assumed in statistics to be symmetric and positive, this function exhibits both the asymmetry and the negative part.

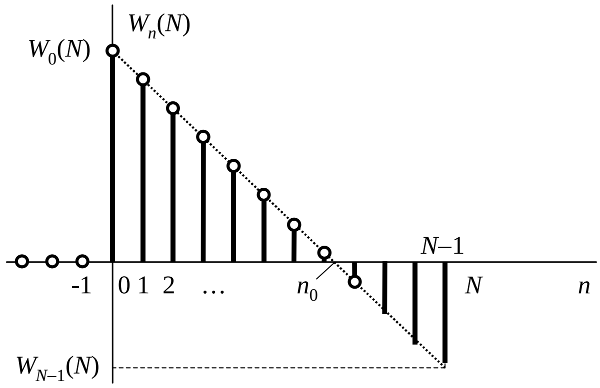

Figure. Weight of an optimally unbiased MA filter

Two special features of the weight (17) may be observed, namely: 1) its area is unity, thus, in terms of DSP this is nothing more than the impulse response of the FIR filter [11]; 2) the maximum-to-minimum ratio of the weight tends to

$$\lim_{N\to\infty}\frac{W_0(N)}{W_{N-1}(N)} = -2\,. \tag{18}$$

This means that the weight may be easily reconstructed if one forgets the formula. We show (Appendix) that the filter (16) is *optimally unbiased*, since the bias is totally compensated for any arbitrary N. The minimum RMSD $\sigma_{OMA}\big|_{N=2} = \sqrt{2}\sigma_{wN}$ corresponds to the minimum number $N = 2$ and, since as N increases, the noise asymptotically tends to $\sigma_{OMA}\big|_{1<<N} = 2\sigma_{wN}$. Now note that even though the bias is totally compensated for, the filter (16) produces noise greater than the simple MA, that is $\sqrt{2}\sigma_{wN} \le \sigma_{OMA} < 2\sigma_{wN}$. Moreover, because the bias is negligible, both the RMSE and the maximum error (A1) have, for N = 2, almost the same minimum magnitude, that is $\approx \sqrt{2}\sigma_{wN}$.

4. GPS-BASED FILTERING OF TIME ERROR GENERATED WITH THE RUBIDIUM CLOCK

Now let us examine the filter (19) for the particular case of time error generated with the rubidium clock[1]. The time error had been measured based on the GPS Timing Receiver of the Motorola Oncore UT+ type with the

[1] Some other cases are examined in [5] and [6]

sample time $\Delta = 100$ sec. To know the trade off, we compared the results to those obtained with the simple MA and three-state and two-state stationary Kalman filters [12], those are the best matched with the time error model (1). Two performances of the filter are of a special interest in timekeeping, these are its error and its time constant (transient). Then, to compare errors, we tune all the filters for the same time constant. Accordingly, we set the proper states of the noises in the signal noise matrix of each Kalman filter, aiming to obtain the nonstationary solution of the Ricatti equation at the level of 0.9 to be equal to the averaging interval. Alternatively, we first design the Kalman filter to be optimal. Then numerically solve the Ricatti equation. Fix its nonstationary solution at the level of 0.9. Finally, determine N for this level [5].

We studied the rubidium clock generated the GPS-based time error with the offset of $y_0 \cong -1.44 \cdot 10^{-12}$. We set N = 80 and obtain the same transient of $\theta = \Delta(N-1) \cong 2.22$ hours for each filter. Because we did not know the origin of the time error x_n^o in (A1) we assumed the three-state Kalman estimate $\hat{x}_n^{3Kal}$ to be the most accurate. Hereupon, substituting $x_n^o = \hat{x}_n^{3Kal}$ for (A1) brings us to the filtering errors. We watch then for good compensation of the bias, which is almost the same as that of the three-state Kalman filter. Though the noise of a simple MA is inherently small, the RMSD of its estimate is bigger than that of the optimally unbiased and two-state Kalman filters, respectively. With this the unbiased filter exhibits less error than that of the two-state Kalman filter. Furthermore, the two-state Kalman filter yields here greater maximum error than that of the optimally unbiased filter.

5. CONCLUSIONS

We have presented and examined in this report the new simple optimally unbiased MA filter (FIR filter) intended especially for the tasks of "on-line" estimation and synchronization in GPS-based timekeeping. The special features of the filter weights (17) and (20) may be observed, these are:

- Unit area, hence, each weight is nothing more than the impulse response of the FIR filter;
- Linear shape with the limited ratio –2 for 1 << N of its positive maximum to its negative minimum (18), thus, this is a simple function.

The special features of the filters (16) and (19) are

- The filter (16) is optimally unbiased and the filter (19) is asymptotically optimally unbiased with the maximum produced noise 2 times bigger than that of a simple MA.
- Both filters yield the same result for 20 ≤ N and (19) is more accurate, since N < 20.

Based on the considered example and the other results [5] and [6], we may now state:

- In practice, the noise of the optimally unbiased filter may be more than 2 times larger than a simple MA. This holds true for a small number of samples n, and since n rises, the noise ratio tends to 2 (Example A1).
- Both filters, (16) and (19), exhibit an intermediate error between those of the three-state and two-state Kalman filters, since all the filters are tuned for the same time constant.

APPENDIX

Express the filtering error as

$$\varepsilon_n = x_n^o - \hat{x}_n , \qquad \text{(A1)}$$

where $\hat{x}_n$ is estimate of a time error, x_n^o is assumed to be an accurate value of a time error. Evaluate (A1) by the particular sample errors for an arbitrary number M of estimates, these are, bias $\Delta\hat{x} = E[\varepsilon_n]$, variance $\sigma_\varepsilon^2 = E[(\varepsilon_n - \Delta\hat{x})^2]$, root-mean-square deviation (RMSD) $\sigma_\varepsilon = \sqrt{\sigma_\varepsilon^2}$, RMS error (RMSE) $\varepsilon_{RMS} = \sqrt{E[\varepsilon_n^2]} = \sqrt{\Delta\hat{x}^2 + \sigma_\varepsilon^2}$, maximum error $\varepsilon_{\max} = \max|\varepsilon_n|$, and global error, which we would like to calculate as $\varepsilon_{\text{global}} = 0.5(\varepsilon_{RMS} + \varepsilon_{\max})$.

Set $x_n^o = x_n$ in (A1) and consider RMSE of the filter, that is

$$\varepsilon_{RMS}^2 = E[\varepsilon_n^2] = E[(x_n - \hat{x}_n)^2] , \qquad \text{(A2)}$$

where x_n changes linearly over the averaging interval, $x_n = \alpha + \beta\Delta n$, α and β are assumed to be expressed by the coefficients of the polynomial (1), and the estimate $\hat{x}_n$ is given by (5) as

$$\hat{x}_n = \sum_{i=0}^{N-1} W_i(N)\xi_{n-i} = o_N^T \mathbf{W}_N , \qquad \text{(A3)}$$

where $o_N(n) = [\xi_n \xi_{n-1} ... \xi_{n-N+1}]^T$ is the observation data vector, of dimensions $N \times 1$, and $\mathbf{W}_N = [W_0(N), W_1(N), ..., W_{N-1}(N)]^T$ is the filter (5) coefficients matrix, of dimensions $N \times 1$. Present (2) for (A3) as $o_N(n) = \mathbf{x}_N(n) + \mathbf{w}_N(n)$, where $\mathbf{x}_N(n) = [x_n x_{n-1} ... x_{n-N+1}]^T$ is the time error data vector, of dimensions $N \times 1$, and $\mathbf{w}_N(n) = [w_n w_{n-1} ... w_{n-N+1}]^T$ is the noise data vector, of dimensions $N \times 1$. Neglect α = 0 (this term does not influence error in the steady state), account $\mathbf{x}_N^T \mathbf{W}_N = \mathbf{W}_N^T \mathbf{x}_N$, and write

$$\varepsilon_{RMS}^2 = E[(\beta\Delta n - \mathbf{x}_N^T \mathbf{W}_N - \mathbf{w}_N^T \mathbf{W}_N)^2]$$

$$\begin{aligned} &= (\beta\Delta n - \mathbf{x}_N^T \mathbf{W}_N)^2 + E[\mathbf{w}_N^T \mathbf{W}_N \mathbf{W}_N^T \mathbf{w}_N] \\ &\quad - 2(\beta\Delta n - \mathbf{x}_N^T \mathbf{W}_N)\mathbf{W}_N^T E[\mathbf{w}_N] \end{aligned}, \quad \text{(A4)}$$

So long as the noise is zero-mean, $E[\mathbf{w}_N] = 0$, then the last terms in (A4) tend to zero. Write o_x ane $= \beta\Delta m n \;\; n - \mathrm{i} \;\; ...n - x \;\; + \mathrm{i}\, 1^n = \beta\Delta t_x$, where $\mathbf{n}_N(n) = [n \;\; n-1 \;\; ...n - N + 1]^T$ is a vector of integers, of dimensions $N \times 1$. Make the transformation $E[\mathbf{w}_N^T \mathbf{W}_N \mathbf{W}_N^T \mathbf{w}_N] = \sigma_w^2 \mathbf{W}_N \mathbf{W}_N^T$, and finally come to the form of

$$\varepsilon_{RMS}^2 = \beta^2 \Delta^2 \left(n - \mathbf{n}_N^T \mathbf{W}_N\right)^2 + \sigma_w^2 \mathbf{W}_N^T \mathbf{W}_N . \quad \text{(A5)}$$

Now deduce:

- The first term in (A5) is a bias of the second order. That means that

$$\Delta\hat{x} = \beta\Delta(n - \mathbf{n}_N^T \mathbf{W}_N) . \quad \text{(A6)}$$

  If the expression in parenthesis of (A6) is equal to zero, $n - \mathbf{n}_N^T \mathbf{W}_N = 0$, the bias compensates in a LMS sense, and the filter is assumed to be *optimally unbiased*;
- The second term in (A5) is the noise variance of the filter

$$\sigma_\varepsilon^2 = \sigma_w^2 \mathbf{W}_N^T \mathbf{W}_N . \quad \text{(A7)}$$

Now examine the filter (16) for the first estimate at the point of $n = N - 1$. Substituting (17) for (A6) shows that it becomes identically zero, so bias is totally compensated and the filter is *optimally unbiased*. Substituting (17) for (A7) yields

$$\sigma_{\varepsilon(16)}^2 = \sigma_{wN}^2 \frac{2(2N-1)}{N+1} , \quad \text{(A8)}$$

where $\sigma_{wN}^2 = \sigma_w^2 / N$ is variance of a discrete noise of a simple MA (3) for the averaging interval of N points. For the minimal number $N = 2$ the formula (A8) produces $\sigma_{\varepsilon(16)}\big|_{N=2} = \sqrt{2}\sigma_{wN}$. With $1 << N$ it becomes $\sigma_{\varepsilon(16)}\big|_{1<<N} = 2\sigma_{wN}$.

Examine the filter (19). The expression in parenthesis of (A6) taking into account (17) yields for $n = N - 1$

$$N - 1 - \mathbf{n}_N^T \mathbf{W}_N \Big|_{1<<N} \cong \frac{3}{N}\Big|_{N\to\infty} \to 0 . \quad \text{(A9)}$$

Thus, the filter is *asymptotically optimally unbiased*. Note, the maximum of $N - 1 - \mathbf{n}_N^T \mathbf{W}_N$ is about 0.48 with $N = 4$, it does not exceed the unit step, and tends to zero since N rises. Substituting (20) for (A7) yields

$$\sigma_{\varepsilon(19)}^2 = \sigma_w^2 \mathbf{W}_N^T \mathbf{W}_N \Big|_{1<<N} \cong 4\sigma_{wN}^2 , \quad \text{(A10)}$$

then $\sigma_{\varepsilon(19)}\big|_{1<<N} = 2\sigma_{wN}$, so the filter produces noise two times larger than the simple MA. Yet, the function (A10) yields $\sigma_{\varepsilon(19)}\big|_{N=2} \cong \sigma_{wN}$ with $N = 2$.

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